Sadlier School

# MATHEMATICAL MINDS IN ACTION:
## What Research Says About Implementing CCSS for Mathematical Practice (SMPs) and NCTM Process Standards

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## Introducion

*Progress® in Mathematics* is a K–5 core mathematics program that situates the teaching and learning of mathematical content within mathematical practices to grow learners' comprehension of both mathematical concepts *and* skills. Being a mathematical thinker and doer requires proficiencies beyond mere mathematical content. Mathematical thinking and doing includes complex skills, such as reasoning, problem solving, modeling, communicating, strategic decision-making, and precise language. These complex skills are the proficiencies and processes of mathematics.

Decades of research have concluded: To become proficient in mathematics, learners need to develop expertise in these complex skills, and our mathematics teaching needs to emphasize and explicitly teach them.

The specificity of mathematical proficiencies and processes have evolved over the last sixty years; yet the emphasis on the ways mathematical learners think about and engage with content and peers has endured. In 1980, the landmark NCTM document, *An Agenda for Action*, established problem solving as a central facet of mathematical teaching and learning. *Adding It Up* (2001) synthesized existing research on mathematics learning to define five interconnected strands of mathematical proficiency (see Figure 1.1). Mathematical proficiency describes the key characteristics of a mathematical thinker and doer and the ultimate goals of mathematical teaching and learning.

The five strands of mathematical proficiency are:

- *conceptual understanding:* comprehension of the concepts, operations, and relations of mathematics,
- *procedural fluency*: knowledge of what mathematical procedures to use, when to use them, and how to use them
- *strategic competence*: the formulation, representation, and solving of mathematical problems by applying conceptual understanding and procedural fluency
- *adaptive reasoning:* logical thought, reflection, explanation, and justification of mathematical concepts and procedures
- *productive disposition:* the belief that mathematics is sensible, useful, and worthwhile and that one is capable of successfully engaging in mathematics (efficacy)

**Figure 1.1** Five Strands of Mathematical Proficiency

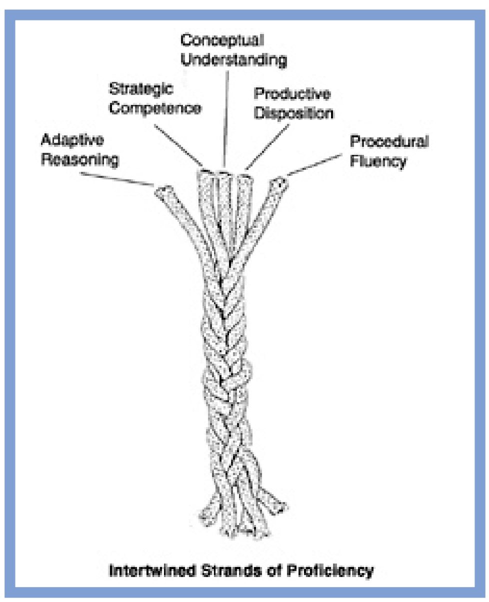



Image from National Research Council's *Adding It Up* (2001), p. 117

NCTM's (2000) *Principles and Standards for School Mathematics* specified five critical topics for mathematical teaching and learning (Content Standards) as well as five critical ways learners should make sense of these topics (Process Standards). NCTM defined the Process Standards as follows:

- *problem solving* is building new knowledge through engaging with and solving problems
- *reasoning and proof* are developing and evaluating mathematical arguments and proof
- *communication* is clearly and coherently talking and writing about mathematical thinking
- *representations* are using multiple representations to model and solve problems
- *connections* are making sense of the relationships among mathematical ideas and real-world situations

The Common Core State Standards for Mathematical Practices (SMPs) (2010) rest on NRC's strands of mathematical proficiency and NCTM's process standards. The SMPs delineate important details about what it looks and sounds like to be a thinker and doer of mathematics and the implications for mathematics teaching and learning. Table 1.1 shows the relationships among the practices, processes, and proficiencies.

**Table 1.1** Alignment of CCSS for Mathematical Practice, NCTM Process Standards, and the Strands of Mathematical Proficiency

| CCSS for Mathematical Practice (SMPs) | NCTM Process Standards | NRC Strands of Mathematical Proficiency |
|---|---|---|
| MP 1 Make sense of problems and persevere in solving them. | Problem Solving<br>Communication<br>Representation | Conceptual Understanding<br>Procedural Fluency<br>Strategic Competence<br>Adaptive Reasoning<br>Productive Disposition |
| MP 2 Reason abstractly and quantitatively. | Reasoning & Proof<br>Problem Solving | |
| MP 3 Construct viable arguments and critique the reasoning of others. | Communication<br>Reasoning & Proof<br>Representation | |
| MP 4 Model with mathematics. | Representation Connections<br>Problem Solving<br>Reasoning & Proof | |
| MP 5 Use appropriate tools strategically. | Problem Solving<br>Representation | |
| MP 6 Attend to precision. | Communication<br>Problem Solving<br>Representation | |
| MP 7 Look for and make use of structure. | Reasoning & Proof<br>Connections<br>Problem Solving | |
| MP 8 Look for and express regularity in repeated reasoning. | Problem Solving<br>Connections<br>Reasoning & Proof | |

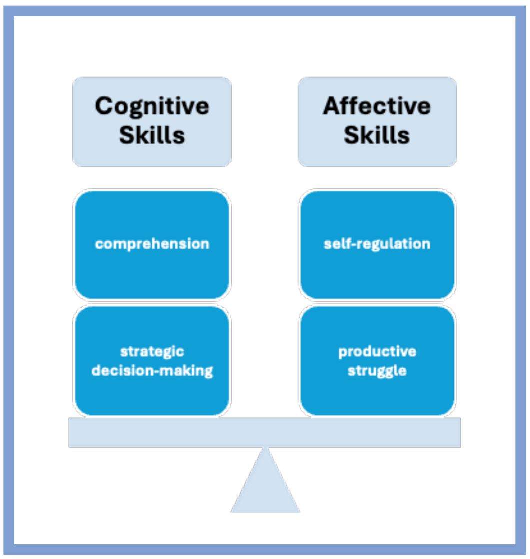
Research in mathematics teaching and learning has continued to deepen our understanding of these processes and proficiencies and effective ways to implement each within our teaching practices (NCTM, 2014 & 2020). This white paper will summarize what we know from research about effective teaching and learning within each of the mathematical practices and show how *Progress in Mathematics* translates this research into teaching practices and resources.

## Make Sense of Problems and Persevere in Solving Them (MP 1)

### What Research Says

The first math practice moves beyond asking learners to provide solutions and answers. It encompasses both cognitive skills and affective skills (Ozturk, Akkan, & Kaplan, 2020) as depicted in Figure 1. To make sense of problems and persevere in solving them requires comprehension, strategic decision-making, self-regulation, and productive struggle all while facing truly problematic situations (Cai & Lester, 2010; Güner & Erbay, 2021; Montague et al., 2014; NCTM, 2020). In other words, learners need to understand what is happening and what is being asked, connect to prior knowledge, plan strategically, select appropriate strategies, self-monitor as they apply those strategies and adjust, and be willing to work through the challenging ups and downs of the process.

**Figure 1.** Defining MP1

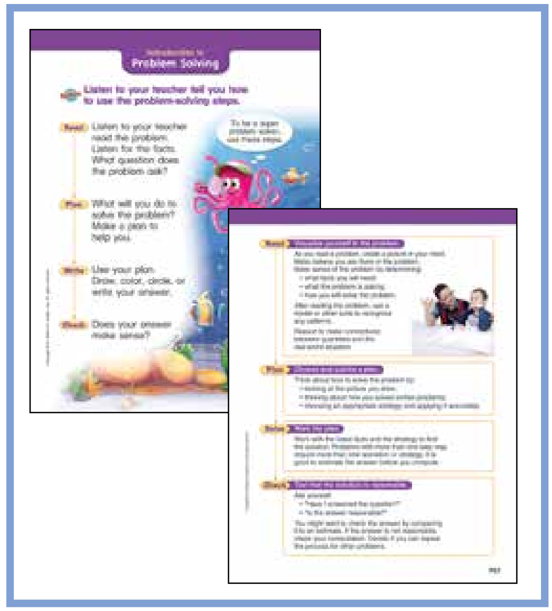



To effectively engage learners in this practice, there are four recommendations for teaching:

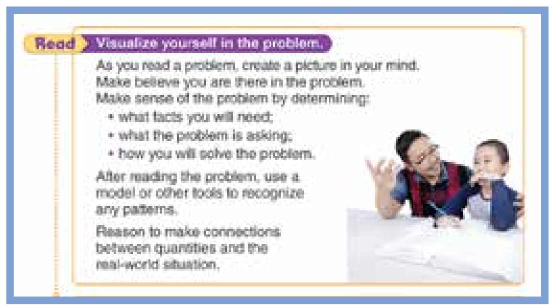
1. Most importantly, **explicitly teach learners to think about their thinking** or their metacognition. This means learners are reflecting on and monitoring their decision-making processes and their understanding of the problem as well as tracking what makes sense, what works, and what does not (Bay-Williams & SanGiovanni, 2021; Chew & Cerbin, 2020; Hattie, 2025; Montague et al., 2014; Woodward et al., 2018). Metacognitive skills in problem-solving are critical.
2. **Explicitly teach multiple problem-solving strategie**s for learners to have a toolbox of appropriate strategies (Bay-Williams & SanGiovanni, 2021; Chew & Cerbin, 2020; Hattie, 2025; Woodward et al., 2018).
3. **Select novel problem**s or problems that learners do not already know how to solve (Barlow et al., 2018; Hattie, 2025; Hiebert & Grouws, 2007).
4. **Explicitly model ways to represent** a novel situation physically, visually, and with abstract mathematical language and notation (Berry & Thunder, 2017; Chew & Cerbin, 2020; Woodward et al., 2018).

### *Within* Progress in Mathematics

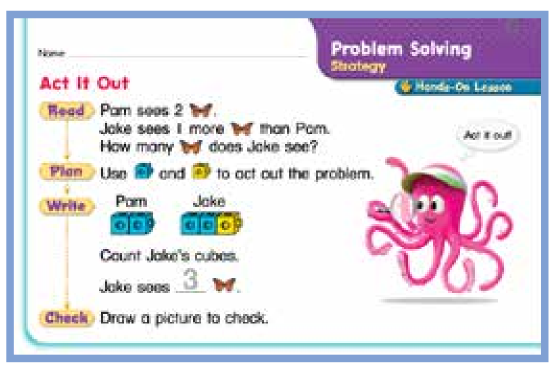
*Progress in Mathematics* implements these four recommendations for effectively engaging learners in making sense of problems and persevering in solving them. Every grade level K–5 begins with a series of explicit Problem-Solving lessons. First, the Four-Step Problem Solving Model is taught: Read, Plan, Solve, and Check. The thinking within each step is made explicit and modeled. **Learners are explicitly taught to think about their thinking.** Kindergarten begins by simply making sense of each step and relying on a metacognitive cue or prompt, such as "Plan: What will you do to solve the problem?" Gradually, each grade level interweaves additional thinking skills within the four steps to develop learners' thinking about their thinking. For example, by Grade 5, the cue for Plan is: "Choose and outline a plan," and the prompts include, "Look at the picture you drew; Think about what you did when you solved similar problems; Choose a strategy or strategies for solving the problem." This metacognitive framework for problem solving is the cornerstone of *Progress in Mathematics*. The four steps create a framework or mnemonic for explicitly teaching and learning about metacognition within problem solving.



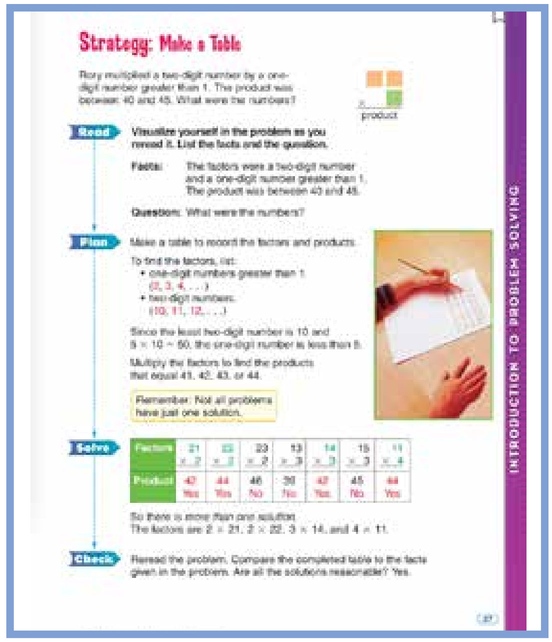

The lessons within the Problem-Solving series also **explicitly teach multiple problem-solving strategies** from Polya's (1945) seminal work. Across Grades K-5, fifteen problem-solving strategies are gradually named, modeled, and practiced, including Find a Pattern, Use Simpler Numbers, and Work Backward. Over time, learners grow a toolbox of strategies that work both broadly across mathematical content strands and deeply within increasingly complex mathematical content. This suite of problem-solving strategies is introduced and then spirally reviewed within and across grades. They become learners' toolbox of strategies, empowering them to be active decision-makers as they strategically select appropriate strategies for novel problems.




There is an explicit lesson to introduce each strategy. These lessons model applying the target strategy within the framework of the Four-Step Problem Solving Process to solve a contextualized task that learners do not already know how to solve or a **novel problem**. The lesson plan includes components to activate prior knowledge, **explicitly teach the problem-solving strategy**, and engage learners in summarizing the new strategy in their Math Journals. There are multitude of novel problems both within the Problem-Solving lesson series and each chapter. **Novel problems** can be contextualized or decontextualized; the essential quality is that they are truly problematic or learners do not readily know how to solve them.



While explicitly teaching the problem-solving strategy, teachers also **model the use of multiple representations**, including concrete, representational, and abstract representations. The representations support activating prior knowledge, each step within the Four-Step Problem Solving Process (Read, Plan, Solve, Check), and learners' summary or generalization of the strategy. Some of the problem-solving strategies themselves are centered upon using multiple representations to visually make sense of problems and solve, such as Act It Out, Draw a Picture, Make a Table, Use a Graph, and Write an Equation. Across chapters, each concept is also supported by multiple representations provided in the Manipulatives Kit, included in the Interactive Problems, and modeled in the colorful graphics of manipulatives and visual representations throughout.

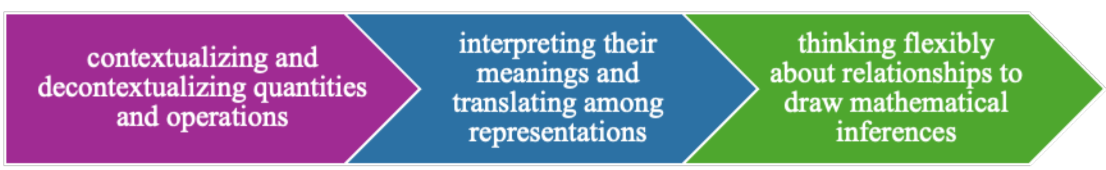



## Reason Abstractly and Quantitatively (MP2)

### What Research Says

To reason abstractly and quantitatively means both contextualizing and decontextualizing quantities and operations, interpreting their meanings, and thinking flexibly about their relationships to draw mathematical inferences (Bay-Williams & SanGiovanni, 2021; Fuchs et al., 2021; Hattie, 2025; NCTM, 2020). As shown in Figure 2, this also requires translating and making connections among multiple representations (Berry & Thunder, 2017; NCTM, 2020). Any novel problem can begin either contextualized (situated within a story or background information) or decontextualized (naked numbers, equations, and expressions). The meaning making happens as learners work to translate the situation into other representations (decontextualizing) or to use language to place the symbols within a meaningful situation (contextualizing). The representations illustrate the meanings and relationships among the quantities and operations, visually show how quantities change when operated upon, and require learners to think flexibly with operations and contexts.

**Figure 2.** Defining MP2.




For example, multiplicative reasoning is understanding how one quantity changes in relation to another. This can be described within a specific context and decontextualized to describe a generalizable relationship or mathematical inference. Reasoning involves thinking abstractly about the relationships between operations as well as quantitatively about the impact of these operations on specific values. There are many types of reasoning in mathematics, including proportional, additive, algebraic, and geometric reasoning.

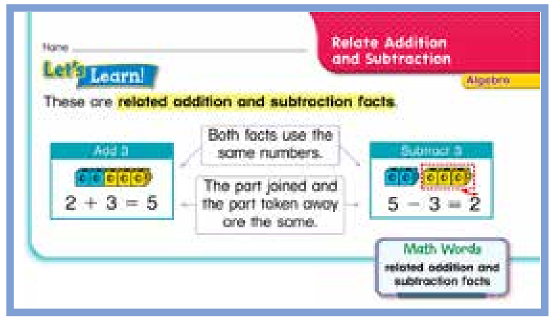
To effectively engage learners in this practice, there are three recommendations for teaching:

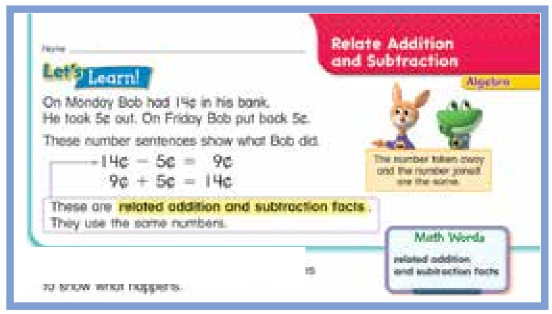
1. **engage in contextualizing and decontextualizing** to focus on making sense of the meaning of quantities and the relationships among operations (Fuchs et al., 2021; Hattie, 2025; Pashler et al., 2007)
2. **make connections among representations** that illustrate these meanings and relationships and visually show how quantities change when operated upon (Berry & Thunder, 2017; Chew & Cerbin, 2020; Flores, Hinton, & Strozier, 2014; Pashler et al., 2007)

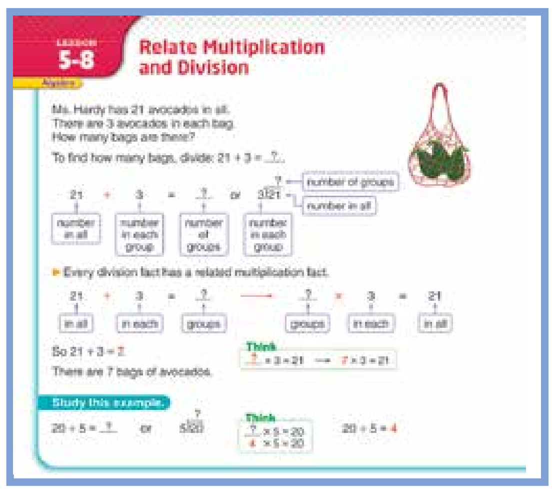
3. **teach related operations together** to focus on thinking flexibly with operations (Bay-Williams & SanGiovanni, 2021; Chew & Cerbin, 2020; Paliwal & Baroody, 2020; Rittle-Johnson, Schneider, & Star, 2015)

### *Within* Progress in Mathematics

*Progress in Mathematics* provides opportunities to engage learners in reasoning abstractly and quantitatively based on these three recommendations.

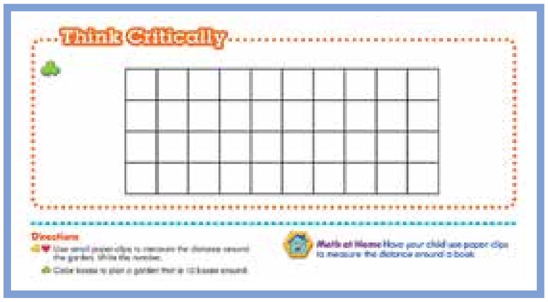


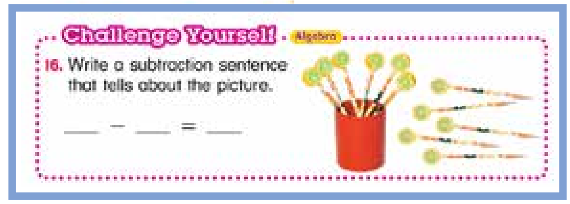



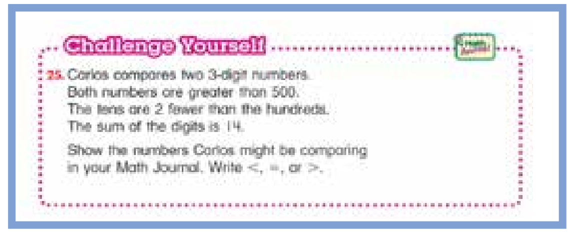




Each lesson begins with a modeled problem. The Teacher's Edition section, Teaching the Lesson, provides teacher-facing support to engage learners in contextualizing and decontextualizing the modeled problem while also translating among multiple representations of the quantities and operations within the problem. In the Student Edition, the section is called Let's Learn. The exemplar task may begin contextualized or decontextualized. Through modeling, guided instruction, and class discussion, learners **engage in contextualizing and decontextualizing**. The focus is on making sense of the meaning of quantities and the relationships among operations. The exemplar illustrates these meanings and relationships through multiple representations (context, visual, symbolic, verbal, and concrete). As learners contextualize and decontextualize the problem, they also **make connections among representations**. Learners visualize and explain the ways quantities change when operated upon.

Further into each lesson, there are additional opportunities to engage in contextualizing and decontextualizing as well as making connections among representations. The Let's Practice section of each lesson in the Student Edition parallels decontextualized tasks called Exercises and contextualized tasks called Problem Solving. The Teacher's Edition section, Practice and Apply, delineates ways to **engage learners in contextualizing and decontextualizing** while **making connections among representations**. This pairing within the Let's Learn tasks provides opportunities for learners to compare the impact of operations on specific values. The End-of-Lesson features, such as Challenge Yourself, Think Critically, and Math Journal ask learners to create their own contexts for operations, select quantities for contexts, describe the meanings of quantities and operations, and think flexibly about their relationships to order to draw mathematical inferences.




At each grade level, there are targeted units and lessons focused on the relationships among operations and thinking flexibly about operations. These units and lessons explicitly **teach related operations together**. For example, in Grades 1 and 2, there are lessons examining the relationships among addition, subtraction, and multiplication of whole numbers. In Grade 3, there are units exploring addition and subtraction of multi-digit numbers as well as the meanings of multiplication and division. The Problem-Solving Strategy Interpret the Remainder emphasizes thinking flexibly about the meaning of the quantity, operations, and context.

## Construct Viable Arguments and Critique the Reasoning of Others (MP3)

### What Research Says

The thinking and doing of mathematics require communication of ideas. Constructing a viable argument translates the reasoning of MP2 into speaking, listening, reading, writing, viewing, and visually representing (Hattie, 2025; NCTM, 2020; Rubin, Estrada, & Honigsfeld, 2022). Critiquing the reasoning of others means learners are responding to each other's thinking in back-and-forth exchanges (Hattie, 2025; NCTM, 2020).

Mathematical discourse increases learners' comprehension of mathematical concepts and skills (Chapin et al., 2009). With a focus on making sense, mathematical discussion clarifies thinking, uncovers misunderstandings, boosts memory, and develops language and social skills while also facilitating peers learning from each other (Chapin et al., 2009; Gresham & Shannon, 2017). Through communication, learners apply multiple strategies to overcome challenges in problem solving and communication (Gresham & Shannon, 2017). Each learner is also held responsible for justifying their reasoning and developing their metacognition (Rawding & Wills, 2012; Walter, 2019).

**Figure 3.** Defining MP3.


To effectively engage learners in this practice, there are three recommendations for teaching:

1. **Establish math talk as a regular routine**. Routines provide guardrails to create social and cognitive safety as well as scaffolds to support learners so that every learner is engaged in the communication of math ideas. (Antonetti & Garver, 2015; Almarode et al., 2024).
2. **Engage in mathematical discussion** or argumentation (Chapin et al., 2009; Gresham & Shannon, 2017; Hattie, 2025; Rawding & Wills, 2012; Walter, 2019).

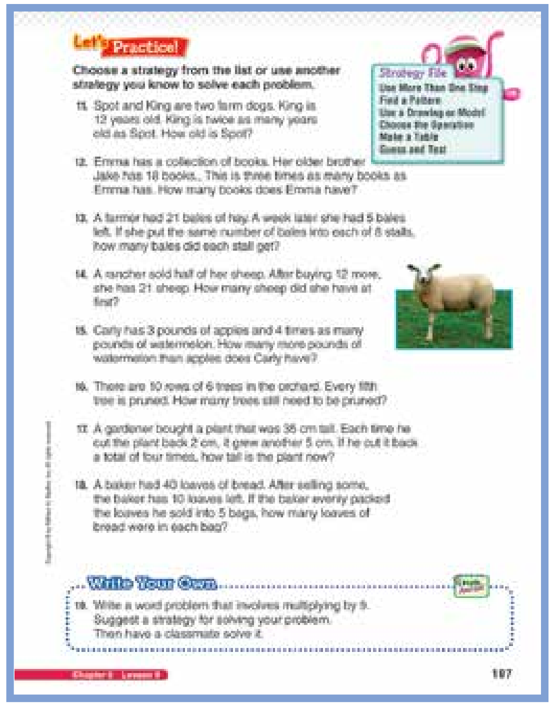
3. **Use multiple modalities** to communicate ideas, including speaking, listening, reading, writing, viewing, and visually representing (Antonetti & Garver, 2015; Almarode et al., 2024; Chew & Cerbin, 2020; Murray et al., 2014; Rubin, Estrada, & Honigsfeld, 2022).

### *Within* Progress in Mathematics

*Progress in Mathematics* translates into teaching practice these recommendations for effectively engaging learners in constructing viable arguments and critiqueing the reasoning of others. There are many components of *Progress in Mathematics* **that establish math talk as a regular routine** so that learners frequently **engage in mathematical discussion** while **using multiple modalities**.

At the beginning of each grade level, the Four-Step Problem Solving Model is reintroduced, new problem-solving strategies are explicitly taught, and then learners are engaged in a mixed review to strategically select, apply, explain, and defend strategies **using multiple modalities**. Learners speak and listen to explain and respond, read and write to decide and defend, and view and visually represent to problem solve and compare strategies. Throughout this problem-solving routine, learners **engage in mathematical discussion**. The Problem-Solving Strategies and Applications creates a routinized way of naming, learning, and selecting strategies while also creating a structure for constructing arguments and critiquing reasoning, thus **establishing math talk as a regular routine**.

STEAM Chapter Openers orient learners to the focal learning objectives of each chapter or unit. The Chapter Openers include a real-world Science Connection based on or supported by an image and a Literature Connection that depicts the math concept or skill within a meaningful context. The Literature Connection is a text, such as a poem or list of statistical information. The Science Connection highlights the mathematical thinking and doing within the text and links it to daily life, such as discerning the number of wings on a butterfly. As a constant component of the curriculum, the Chapter Openers establish math talk as a regular routine where learners engage in mathematical discussion as they make sense of each connection. The Science Connection, images, and mathematical discourse provide opportunities to use multiple modalities as learners make sense of the chapter's focal learning objectives.


Let's Practice!

Choose a strategy from the list or use another strategy you know to solve each problem.

Strategy File
Use More Than One Step
Find a Pattern
Use a Drawing or Model
Choose the Operation
Make a Table
Guess and Test

11. Spot and King are two farm dogs. King is 12 years old. King is twice as many years old as Spot. How old is Spot?
12. Emma has a collection of books. Her older brother Jake has 18 books. This is three times as many books as Emma has. How many books does Emma have?
13. A farmer had 21 bales of hay. A week later she had 5 bales left. If she put the same number of bales into each of 8 stalls, how many bales did each stall get?
14. A rancher sold half of her sheep. After buying 12 more, she has 21 sheep. How many sheep did she have at first?
15. Carly has 3 pounds of apples and 4 times as many pounds of watermelon. How many more pounds of watermelon than apples does Carly have?
16. There are 10 rows of 6 trees in the orchard. Every fifth tree is pruned. How many trees still need to be pruned?
17. A gardener bought a plant that was 35 cm tall. Each time he cut the plant back 2 cm, it grew another 5 cm. If he cut it back a total of four times, how tall is the plant now?
18. A baker had 40 loaves of bread. After selling some, the baker has 10 loaves left. If the baker evenly packed the loaves he sold into 5 bags, how many loaves of bread were in each bag?

Write Your Own

19. Write a word problem that involves multiplying by 9. Suggest a strategy for solving your problem. Then have a classmate solve it.

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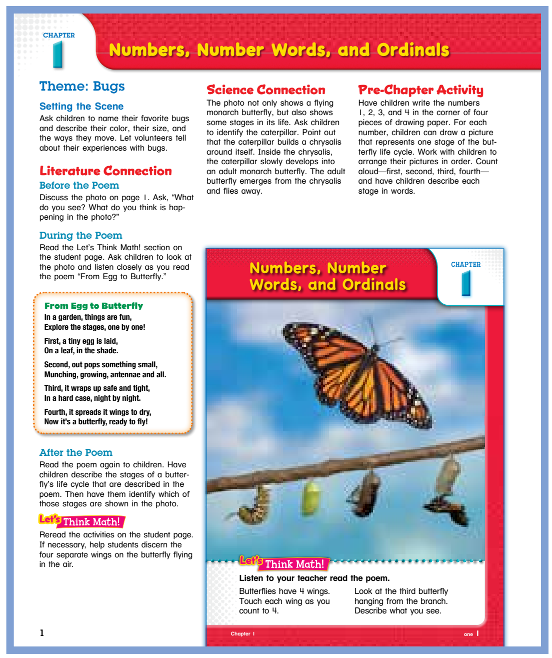
CHAPTER 1 Numbers, Number Words, and Ordinals

Theme: Bugs

Setting the Scene
Ask children to name their favorite bugs and describe their color, their size, and the ways they move. Let volunteers tell about their experiences with bugs.

Literature Connection
Before the Poem
Discuss the photo on page 1. Ask, "What do you see? What do you think is happening in the photo?"

During the Poem
Read the Let's Think Math! section on the student page. Ask children to look at the photo and listen closely as you read the poem "From Egg to Butterfly."

From Egg to Butterfly
In a garden, things are fun,
Explore the stages, one by one!
First, a tiny egg is laid,
On a leaf, in the shade.
Second, out pops something small,
Munching, growing, antennae and all.
Third, it wraps up safe and tight,
In a hard case, night by night.
Fourth, it spreads it wings to dry,
Now it's a butterfly, ready to fly!

After the Poem
Read the poem again to children. Have children describe the stages of a butterfly's life cycle that are described in the poem. Then have them identify which of those stages are shown in the photo.

Let's Think Math!
Reread the activities on the student page. If necessary, help students discern the four separate wings on the butterfly flying in the air.

Science Connection
The photo not only shows a flying monarch butterfly, but also shows some stages in its life. Ask children to identify the caterpillar. Point out that the caterpillar builds a chrysalis around itself. Inside the chrysalis, the caterpillar slowly develops into an adult monarch butterfly. The adult butterfly emerges from the chrysalis and flies away.

Pre-Chapter Activity
Have children write the numbers 1, 2, 3, and 4 in the corner of four pieces of drawing paper. For each number, children can draw a picture that represents one stage of the butterfly life cycle. Work with children to arrange their pictures in order. Count aloud—first, second, third, fourth—and have children describe each stage in words.

Numbers, Number Words, and Ordinals CHAPTER 1

Let's Think Math!
Listen to your teacher read the poem.
Butterflies have 4 wings. Touch each wing as you count to 4.
Look at the third butterfly hanging from the branch. Describe what you see.

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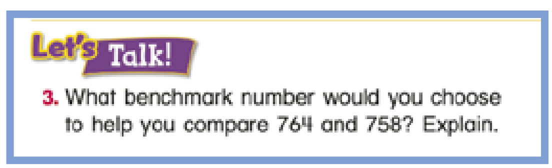

Let's Talk!

3. What benchmark number would you choose to help you compare 764 and 758? Explain.

The Let's Talk! section is a regular feature of daily lessons, thus **establishing math talk as a regular routine**. In the *Let's Talk!* section of every lesson, there are mathematical discourse prompts. These prompts encourage learners to draw inferences, pose conjectures, make connections, and inductively reason. Whether in whole group, small group, or partner discussion, the Let's Talk! questions **engage learners in mathematical**

**discourse** or argumentation. Other routines across every lesson, such as the End-of-Lesson Features, the Summarize/Assess section of each lesson, the *Interactive Problem of the Day*, and *Interactive Mental Math*, center around constructing arguments, critiquing reasoning, and documenting thinking. These are powerful routines that engage learners in **using multiple modalitie***s* to make sense of each lesson's mathematical content. Multiple formative assessment tools, such as entry and exit tickets, observational assessments, and *Check Your Progress*, capture learners' understanding of concepts and skills as they communicate their reasoning in multiple modalities.

## Model with Mathematics (MP4)

### What Research Says

Modeling is what makes mathematics unique. It is engaging in the process of mathematical modeling or using mathematical tools to represent, analyze, and solve contextualized problems (Eck, Garcke, & Knabner, 2017; NCTM, 2020). Mathematical modeling (see Figure 4) involves decontextualizing or abstracting complex contexts into simplified mathematical representations, describing the quantitative relationships and changes using abstract representations and tools, and then analyzing the relationships to make predictions and apply the model within the context (Eck, Garcke, & Knabner, 2017; NCTM, 2020). This is the heart of mathematical thinking and doing.

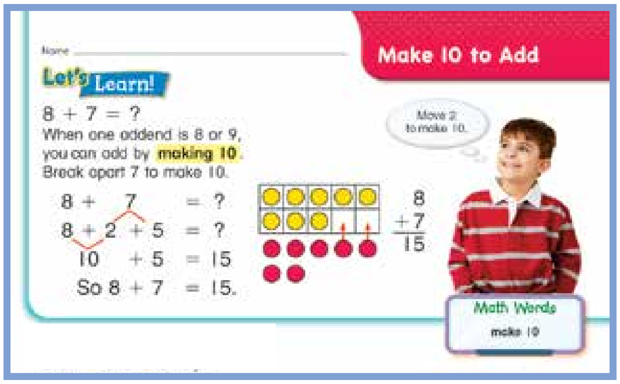
**Figure 4.** Defining MP4.

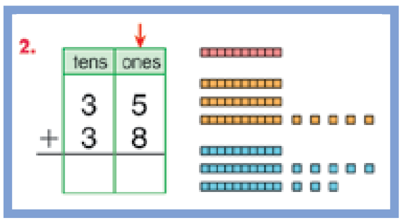

Many of the research-based recommendations for engaging in the mathematical practices overlap because the practices interweave to describe practical ways to grow thinkers and doers of mathematics. This means many of the recommendations for effectively engaging learners in the previous and subsequent mathematical practices apply to modeling with mathematics. But there are two unique research recommendations:

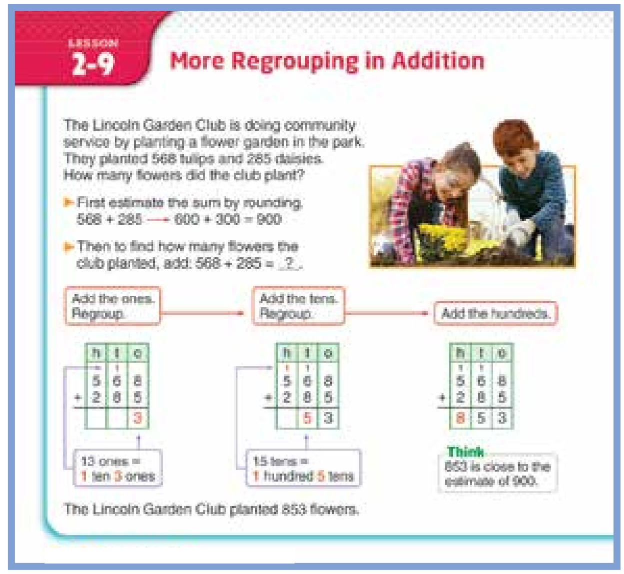
1. **Implement the CRA** (concrete-representational-abstract) **method** to explicitly teach multiple representations and tools for mathematical modeling. Research across grade levels and with all populations of learners (including learners with special needs and English Language Learners) has demonstrated that CRA is an effective teaching strategy (Chew & Cerbin, 2020; Crawford & Ketterlin-Geller, 2008; Fuchs et al., 2021; Hudson, Miller, & Butler, 2006; Paulsen, 2005; Witzel, 2005). Our teaching should use parallel modeling of each level of representation and engage learners in connecting the representations as well as talking about their thinking about the representations.
2. **Explicitly model using five types of representations:** physical (tangible objects, acting out, manipulatives), visual (drawings, pictures, images, charts, graphs), verbal (mathematical language and vocabulary), contextual (real-world situations, problems embedded in content or texts), and symbolic (mathematical symbols, including algebraic letters) (NCTM, 2014; Ünal et al, 2023).

### *Within* Progress in Mathematics

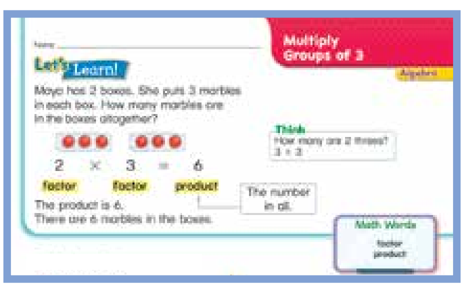
*Progress in Mathematics* emphasizes modeling with mathematics and the two unique recommendations for implementing this practice. Each lesson includes concrete, representational (or pictorial), and abstract representations of the concept. The Teacher's Edition section, *Teaching the Lesson,* details how to **implement the CRA method**, including parallel modeling of the lesson's content using concrete, pictorial, and abstract representations and engaging learners in thinking and talking about the connections among the representations. The Student Edition includes cues, such as "Use cubes to model each exercise." colorful graphics as images of the concrete representations, and examples of abstract representations, such as equations and expressions. Learners move fluidly between representations and are prompted in the Let's Talk! and Let's Write! sections to reflect on the connections.

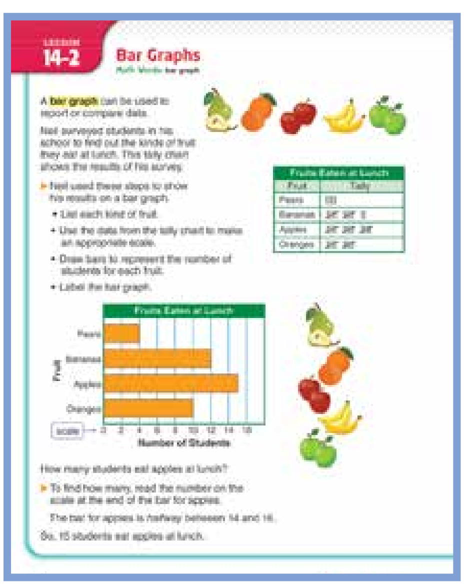


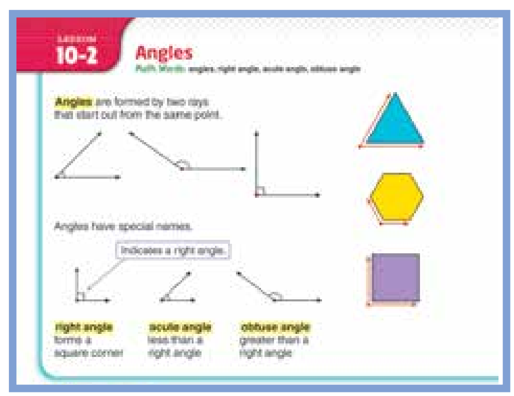



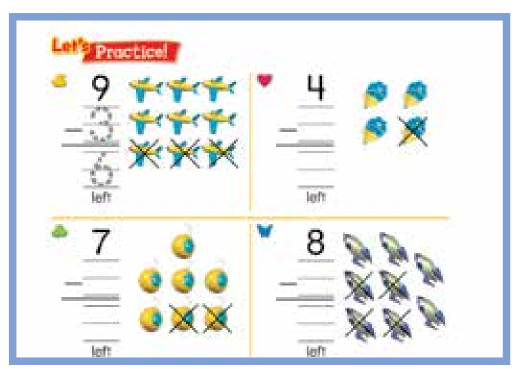




Each of the five types of representations are included across the units and the Teacher's Edition delineates ways to **explicitly model using five types of representations**. The manipulatives kit includes a variety of physical representations. The graphics include images of the manipulatives and physical objects. Learners interpret and construct additional visual representations, such as drawings, ten-frames, charts, number lines, and graphs. The Problem Solving lessons and End-of-Lesson features, such as Challenge Yourself and Problem Solve, include contextual representations integrated with visual and symbolic representations. The Let's Learn! and Let's Practice! sections feature symbolic representations with prompts and cues to translate these into physical, visual, and verbal representations. The Let's Talk! and Let's Write! sections as well as the Math Words text boxes support verbal representations.



## Use Appropriate Tools Strategically (MP5)

### What Research Says

Using appropriate tools strategically is about being a decision-maker. Learning the problem-solving strategies of MP1 and the representations of MP4 adds tools to learners' toolboxes. As shown in Figure 5, this practice focuses on developing learners' strategic selection and application of appropriate tools (strategies and representations) for the situation (Bay-Williams & SanGiovanni, 2021; NCTM, 2020; Woodward et al., 2018).

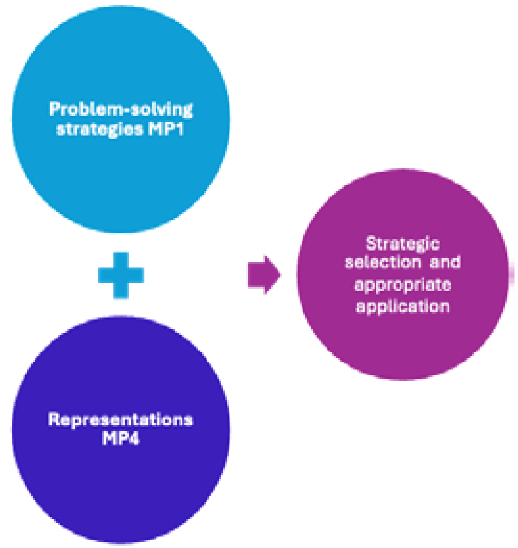


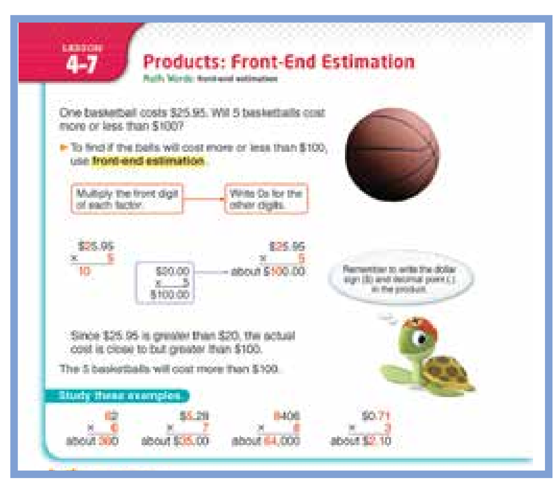

**Figure 5.** Defining MP5.

To effectively engage learners in using appropriate tools strategically, educational research provides three recommendations:

1. **Use estimation to monitor problem solving strategies and solutions**. Rather than teaching estimation as a separate skill, estimation should be an active decision-making tool before, during, and after problem solving in order to determine if the tools were appropriate and the solution accurate (Andrews et al, 2021; Bay-Williams & SanGiovanni, 2021).
2. **Explicitly teach metacognitive processes to evaluate the appropriateness of a tool**, including a mental math strategy, a manipulative, a calculator, and other technology (Bay-Williams & SanGiovanni, 2021; Chew & Cerbin, 2020; Fuchs et al., 2021; Hattie, 2025; Woodward et al, 2018).
3. **Deliberately practice strategic tool selection** (Almarode et al., 2024; Chew & Cerbin, 2020; Fuchs et al., 2021; Hattie, 2025; Pashler et al., 2007; Woodward et al, 2018). Deliberate practice is when an individual works on an exercise or task with a specific goal for learning (such as evaluating the appropriateness of a tool or using estimation to evaluate a strategy and solution) (Almarode et al., 2024; Ericsson & Pool, 2016). The learner receives effective feedback and develops a mental model for monitoring appropriate tool selection in the future.

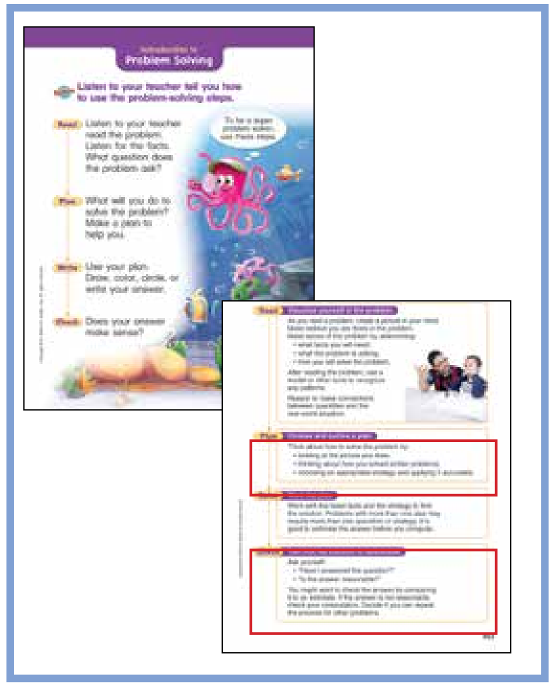
### *Within* Progress in Mathematics

*Progress in Mathematics* provides opportunities to enact these three recommendations for effectively developing learners' strategic use of appropriate tools. Grades 1 through 5 balance an emphasis on explicitly teaching estimation strategies for operating with whole and rational numbers with applying those strategies to monitor problem solving strategies and solutions. For example, there are lessons on estimating sums and differences of whole numbers (Grades 1–4), estimating products of whole numbers (Grade 4), estimating quotients of whole numbers (Grade 5), and estimating sums and differences of fractions and decimals (Grade 5). One of the strategies in the suite of Problem-Solving Strategies is Guess and Test/Check, which involves using estimation to predict and monitor the accu-



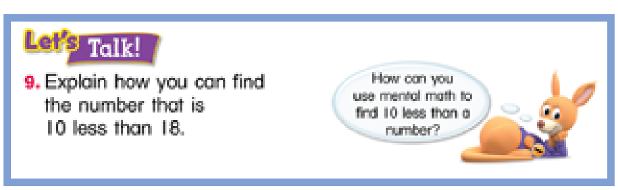
racy of potential solutions. Once estimation strategies are taught explicitly, the application of those strategies spirals to emphasize the **use of estimation to monitor problem-solving strategies and solutions**.

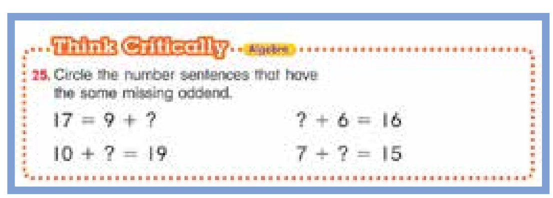
*Progress in Mathematics* **explicitly teaches metacognitive processes to evaluate the appropriateness of a tool**, using the Four-Step Problem-Solving Process and additional cues and prompts within each section of the lesson. The Four-Step Problem-Solving Process (Read, Plan, Write, Check) is a mnemonic or framework to monitor the metacognitive processes of problem solving. The second step, Plan, involves choosing one or more strategies, comparing the current problem to previously solved problems, and making an estimate. The fourth step, Check, involves looking back to determine if the answer or solution is reasonable, **using estimation to monitor the accuracy of the strategy and solution**, and to evaluate whether the appropriate tools were applied.




There are also prompts and cues to engage in metacognitive reflection about tool selection throughout each lesson. Within Let's Learn!, Let's Practice!, and Let's Talk!, there are prompts included in the directions, such as "Ask yourself, 'Does my answer make sense? How can I check my answer?'" Similar questions to evaluate the appropriateness of tools are included in End-of-Chapter features, such as Challenge Yourself and Think Critically.

At the beginning of each grade level, learners add new strategies to their toolbox and then **deliberately practice strategic tool selection** in a mixture of contextualized problems, called Problem-Solving Applications: Mixed Review. Within every lesson, the Let's Practice! section engages learners in deliberate practice focused on the goal of the lesson. Here, learners deliberately practice strategic tool selection and are prompted within End-of-Lesson features to reflect on their tool selection. There are also individual lessons that explicitly teach a mental model for monitoring appropriate tool selection, **deliberately practice tool selection**, and provide effective feedback.



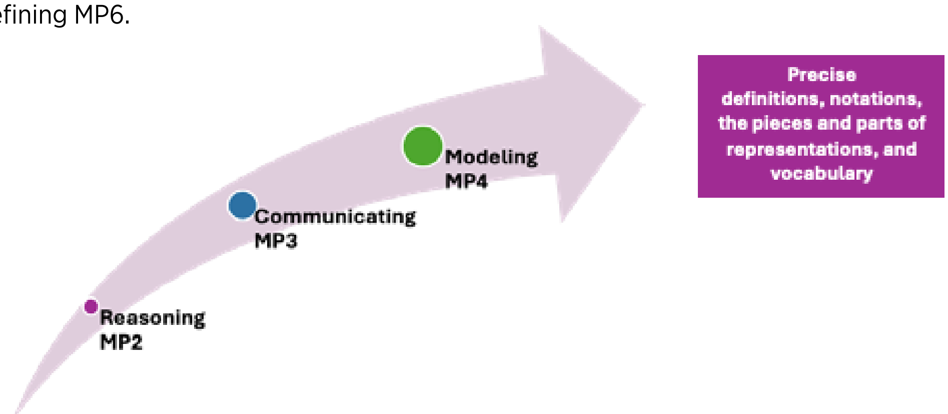



## Attend to Precision (MP6)

### What Research Says

The mathematical practice of attending to precision is not simply about accuracy; it is first and foremost about attending to details and communicating those details in speaking, writing, and representing. As shown in Figure 6, the practices of reasoning, communicating, and modeling intersect here to emphasize the importance of attending to precise definitions, notations, the pieces and parts of representations (such as graphs and number lines), and vocabulary (Almarode et al., 2024; NCTM, 2020).

**Figure 6.** Defining MP6.

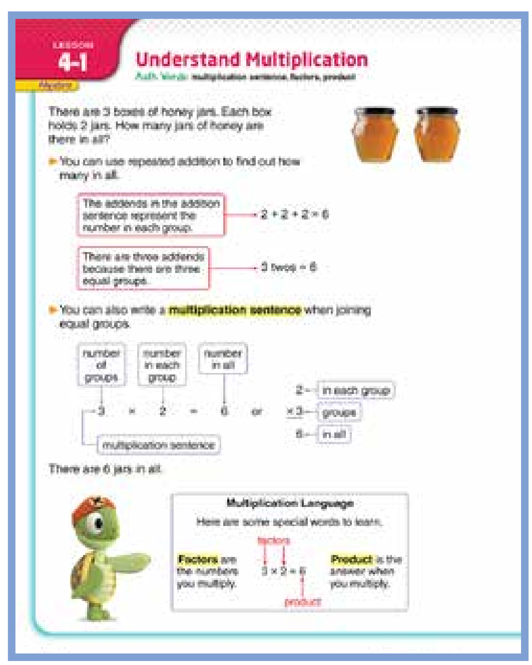


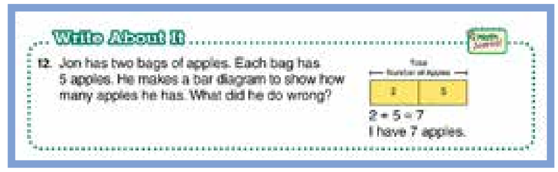

To effectively engage learners in attending to precision, there are two significant recommendations:

1. **Analyze worked examples**, which includes correct, partially correct, and incorrect worked examples (Almarode et al., 2024; Bay-Williams & SanGiovanni, 2021; Chew & Cerbin, 2020; Hattie, 2025; Pashler et al., 2007). Emphasize and teach the pieces and parts of mathematical models, the notation, and vocabulary within the context of the worked examples.
2. **Explicitly teach mathematical language and notation** with an emphasis on making meaning through speaking, listening, reading, writing, viewing, and visually representing (Fuchs et al., 2021; Hansen & Thunder, 2014; Hattie, 2025; Rubin, Estrada, & Honigsfeld, 2022; Woodward et al, 2018).

### *Within* Progress in Mathematics

Progress in Mathematics translates into practice these two recommendations for effectively engaging learners in attending to precision. Within every Let's Learn! section of a lesson, there are opportunities to engage learners in **analyzing worked examples**. The Let's Learn! section includes guided instruction that pairs visual representations with mathematical notation and vocabulary. There are annotations within the worked examples to emphasize the important models, notation, and vocabulary. Additionally, there are opportunities to **analyze worked examples** as learners begin to work independently within each set of exercises. Within the suite of problem-solving strategies, the strategy Use Simpler Numbers/Problems teaches learners to make use of creating and analyzing a simpler worked example in order to solve a parallel but more complex problem.







There is a plethora of resources for **explicitly teaching mathematical language and notation**. Embedded within each lesson, the Student Edition includes *Math Words*, yellow highlighted vocabulary, and images and examples of the mathematical language. The Teacher's Edition provides explicit vocabulary instructional support to engage learners with the mathematical language before and while using the Student Edition page. For example, learners may use concrete manipulatives to enact and make connections with the images on the page or they may compare and contrast two mathematical representations using mathematical language. Combined with Let's Talk! and the End-of-Lesson features, these components emphasize making meaning of mathematical language and notation through multimodalities.

Math Vocabulary Flash Cards, Audio/Visual Glossary, and Multilingual Learner Support offer digital resources for both teachers and learners to make sense of mathematical vocabulary and precise notation.

## Look For and Make Use of Structure (MP7)

### What Research Says

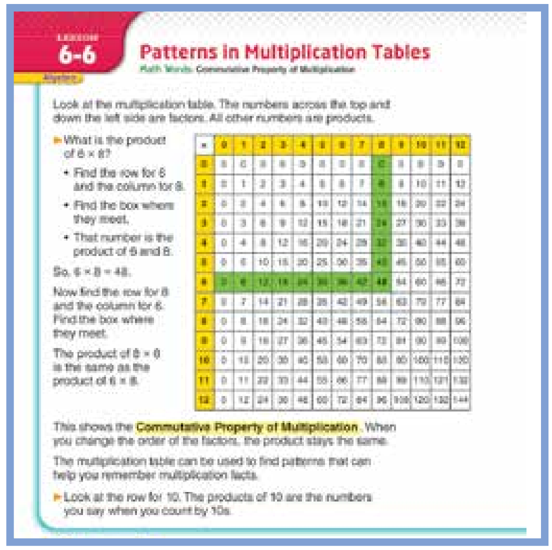
While the number system is ever expanding, it is also highly structured and full of patterns, properties, and relationships. This is the inspiration for the math practice, look for and make use of structure. This practice means mathematical thinkers and doers are constantly seeking patterns, equivalences, and relationships within the number system (Bay-Williams & SanGiovanni, 2012; NCTM, 2020). They are pursuing, finding, and applying order and structure. The figure below represents this practice.

**Figure 7.** Defining MP7.

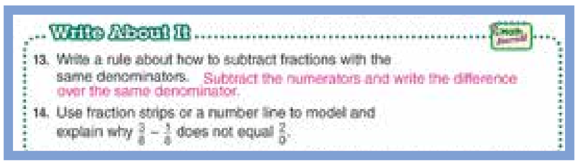

To effectively engage learners in looking for and making use of structure, there are two recommendations:

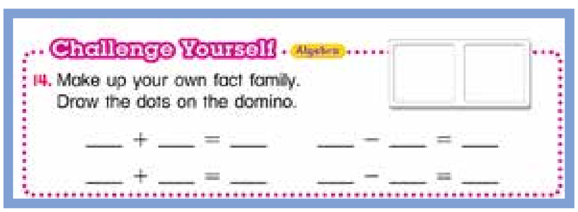
1. **Emphasize patterns, equivalences, and connections** rather than isolated ideas and rules that expire (Chew & Cerbin, 2020; Hattie, 2025; Karp, Bush, & Dougherty, 2014; Sun, Sun, & Xu, 2023).
2. **Engage even the youngest learners in algebraic and functional thinking** (Blanton et al., 2015; Blanton et al., 2017; Sun, Sun, & Xu, 2023). This means all learners should be engaged in thinking about functional relationships, variable quantities and variable notation within meaningful and developmentally appropriate contexts.

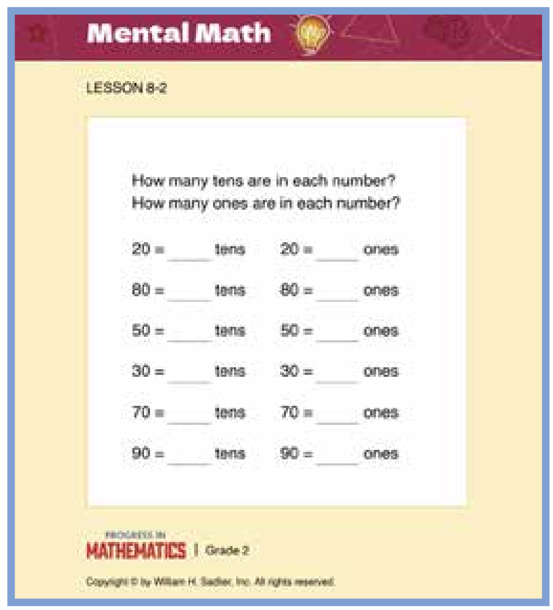
### *Within* Progress in Mathematics




*Progress in Mathematics* provides opportunities to bring to life these two recommendations for effectively engaging learners in looking for and making use of structure. Every grade level includes focused lessons on mathematical patterns and representing values and contexts in equivalent ways. Both the Teacher and Student Editions **emphasize patterns, equivalences, and connections** during the Let's Learn! portion of these lessons with prompts like, "Look for a pattern when counting," and "What division sentence is modeled by the same figure?" Then, in Let's Talk! and the End-of-Chapter features, there are additional prompts throughout lessons that emphasize pursuing, finding, and applying patterns, equivalences and relationships within the number system. For example, Do You Remember? prompts learners to make connections to prior learning through purposefully related spiral review tasks. Think Critically, Challenge Yourself, and Problem Solve prompt learners to look for and make use of a pattern, equivalence, or connection.



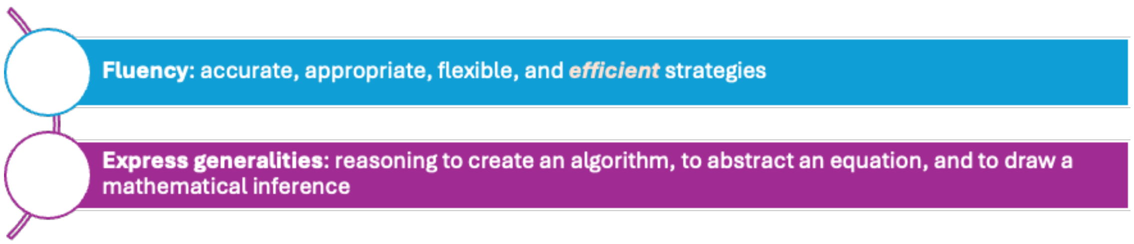

Similarly, there are focused lessons labeled "Algebra," which promote the relationship of numbers and operations. These lessons begin in Kindergarten so that even our youngest learners engage in algebraic and functional thinking. Across all lessons, there are also opportunities to explicitly and implicitly engage in algebraic and functional thinking. For example, in the Challenge Yourself sections of lessons, there are explicit prompts to encourage learners to reflect on and make connections between relationships, quantities, and notation. The Let's Learn! section of graphing lessons engages learners in comparing data representations to describe how the data varies across graphs. The Let's Practice section includes contextualized problems, where values are dependent on each other and unknown values can be represented using abstract symbols. Implicitly, the simple representation of some words with images within the contextualized problems of Let's Learn! and Let's Practice! also encourages the youngest learners to routinely make sense of the concept of variable quantities and notation.

## Look For and Express Regularity in Repeated Reasoning (MP8)

### What Research Says

Looking for and expressing regularity in repeated reasoning is the math practice that develops learners who are fluent and able to express generalities (see Figure 8). Becoming fluent means finding accurate, appropriate, flexible, and most important to this practice, *efficient* strategies (Bay-Williams & SanGiovanni, 2021; NCTM, 2020). A generalizable process requires reasoning to express an algorithm, to abstract an equation, and to draw a mathematical inference (Bay-Williams & SanGiovanni, 2021; Bay-Williams & Stokes Levine, 2017; NCTM, 2020; Sun, Sun, & Xu, 2023).

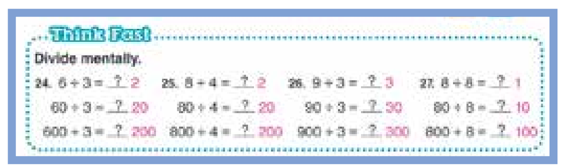
**Figure 8.** Defining MP8.




To effectively engage learners in looking for and expressing regularity in repeated reasoning, there are two recommendations unique from the previous practices:

1. **Pose questions that lead to generalizations and mathematical inferences** (Bay-Williams & Stokes Levine, 2017; Chew & Cerbin, 2020; Hattie, 2025; Rittle-Johnson, Schneider, & Star, 2015). Ask questions like, "Will it always work? What if?" to orient learners to transfer, application and generalization of strategies.

2. **Explicitly teach generalizable strategies that are efficient** (Bay-Williams & SanGiovanni, 2021; Chew & Cerbin, 2020). There are seven significant reasoning strategies and seven automaticities that support a number sense-based approach to fluency (Bay-Williams & SanGiovanni, 2021). Teaching these explicitly and engaging learners in deliberate practice reduces learners' cognitive load and expands learner's access to problem solving in novel situations.




### *Within* Progress in Mathematics

Progress in Mathematics provides opportunities to take action on these two recommendations for effectively engaging learners in looking for and expressing regularity in repeated reasoning. Within the Student Edition, learners are prompted to pause and consider generalizations and mathematical inferences through open questions in the Let's Learn!, Let's Talk! and End-of-Chapter sections. The Math Journal is frequently a tool for documenting these complex ideas. Within the Teacher's Edition, additional open questions aligned with the lesson's objective are included. There are also supports for **posing questions that lead to generalizations and mathematical inferences**, especially implementation strategies that allow for learners to productively struggle and share multiple perspectives.

The Problem-Solving lessons **explicitly teach generalizable strategies that are efficient**, specifically fifteen of Polya's problem solving strategies. Throughout each chapter and lesson, there are many opportunities for learners to engage in deliberate practice of these efficient strategies in order to develop fluency and make sense of generalizable strategies. Each lesson has a Let's Practice! section that includes both exercises and problems.

Within chapters focused on operation sense, there are lessons that explicitly teach generalizable strategies that are efficient for each operation with whole and rational numbers. For example, in Grade 2, there is a series of lessons for addition of whole numbers that includes the strategies: related addition facts, count on to add, extend facts to 20, make 10 to add, doubles facts, doubles +1, doubles -2, add ones and tens, and compensation. Also in Grade 2, there is a series of lessons for subtraction of whole numbers that includes the strategies: count up, make 10, count back, and think addition.

# Contents

CHAPTER 1

## Addition and Subtraction Facts

Theme: Animals



★ Lesson promotes algebraic reasoning.

v

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